# Oscillation of Third-Order Mixed Type Nonlinear Neutral Differential Equations

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***Abstract*- Sufficient conditions for the oscillation and asymptotic behavior of solutions of neutral differential equations of the form**

$$\left[a(t)\left[\left(x(t)+c(t)x(\delta(t))\right)''\right]^{\gamma}\right]'=q(t)x^{\alpha}(\tau(t))+p(t)x^{\beta}(\sigma(t))$$

**are established using comparison theorems. They improve and complement to the existing results. Examples are provided to illustrate the main results.**

***Keywords*- *Third Order Neutral Differential Equations; Mixed Arguments; Oscillation***

## I. INTRODUCTION

This paper deals with the oscillatory and asymptotic behavior of solutions of the third order neutral differential equation

$$\left[a(t)\left[\left(x(t)+c(t)x(\delta(t))\right)''\right]^{\gamma}\right]'=q(t)x^{\alpha}(\tau(t))+p(t)x^{\beta}(\sigma(t)) \quad (1.1)$$

subject to the following conditions :

($C_1$) $a(t),\ c(t),\ q(t),\ \tau(t),\ p(t),\ \sigma(t),\ \delta(t)$ are real continuous functions defined for $t \ge t_0 \ge 0$;

($C_2$) $\alpha,\beta,\gamma$ are ratio of odd positive integers;

($C_3$) $a(t),\ q(t),\ p(t)$ are positive and $0 \le c(t) \le c < 1$;

($C_4$) $\delta(t) \le t,\ \tau(t) \le t,\ \sigma(t) \ge t,\ \tau(t)$ and $\sigma(t)$ are nondecreasing with $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty$;

($C_5$) $$\int_{t_0}^{\infty}\frac{1}{a^{\frac{1}{\gamma}}(t)}dt=\infty.$$

We set $z(t)=x(t)+c(t)x(\delta(t))$. By a solution of equation (1.1), we mean a function $x(t)\in C([T_x,\infty)), T_x \ge t_0$, which has the properties $z(t)\in C^2([T_x,\infty)), a(t)(z'(t))^{\alpha}\in C'([T_x,\infty))$ and satisfies equation (1.1) on $[T_x,\infty)$. We assume only those solutions $x(t)$ of equation (1.1) which satisfy $\sup\{|x(t)|: t\ge T\}>0$ for all $T \ge T_x$, and assume that equation (1.1) possesses such a solution. A solution of equation (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise, it is called nonoscillatory.

The problem of determining oscillation and nonoscillation of solutions of functional differential equations received a great interest in recent years, see for example the monographs [5, 7] and the references contained therein. For the special case of equation (1.1) when $c(t)=0$ and either $p(t)\equiv 0$ or $q(t)\equiv 0$ are studied in [2-4, 8]. In [1, 6] the authors studied the oscillatory behavior of equation (1.1) when $c(t)\equiv 0$.

Motivated by the above observation in this paper we study the oscillatory and asymptotic behavior of solutions of equation (1.1) .So our results complement and extend the earlier ones presented in [1,6]. Examples are provided to illustrate the main results.

## II. PRELIMINARY RESULTS

First we present comparison results which will be used to prove our main results.

**Lemma 2.1.** Assume $p(t), \sigma$, and $\beta$ satisfy conditions ($C_3$), ($C_4$) and ($C_2$) respectively. If the first order advanced differential inequality $u'(t) - p(t)u^{\beta}(\sigma(t)) \ge 0$ has eventually positive solution then so does the advanced differential equation $u'(t) - p(t)u^{\beta}(\sigma(t)) = 0$.

Proof. The proof is similar to that of Lemma 1 in [1] and hence the details are omitted.

Next we classify all possible nonoscillatory solutions of equation (1.1) when condition ($C_5$) is satisfied.

**Lemma 2.2.** Let $x(t)$ be a nonoscillatory solution of equation (1.1). Then $z(t)=x(t)+c(t)x(\delta(t))$ satisfies eventually, one of the following cases:

(I) $z(t)z'(t) > 0;\ z(t)z''(t) > 0;\ z(t)\left[a(t)(z''(t))\right]' > 0;$

(II) $z(t)z'(t) > 0;\ z(t)z''(t) < 0;\ z(t)\left[a(t)(z''(t))\right]' > 0.$

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0,\ x(\delta(t)) > 0$ for all $t \ge t_1 \ge t_0$. Then $z(t)>0$ and $\left[a(t)(z''(t))\right]'>0$, eventually. Thus $z'(t)$ and $z''(t)$ are of fixed sign for $t \ge t_2$, $t_2$ is large enough.

At first we assume that $z''(t)<0$. Then either $z'(t)>0$ or $z'(t)<0$, eventually. But $z''(t)<0$ together with $z'(t)<0$ implies $z(t)<0$, a contradiction, that is, case (II) holds.

Now suppose that $z''(t)>0$, then either $z'(t)>0$ or $z'(t)<0$ holds. On the other hand, if ($C_5$) holds, then $\left[a(t)\left(z''(t)\right)^{\gamma}\right]'\geq 0$ implies that $a(t)\left(z''(t)\right)^{\gamma}\geq d>0$,

$t\geq t_2$. Integrating from $t_2$ to $t$, we obtain,

$$z'(t)-z'(t_1)\geq d^{\frac{1}{\gamma}}\int_{t_2}^{t}\frac{1}{a^{\frac{1}{\gamma}}(s)}ds.$$

which implies that $z'(t)\to\infty$ as $t\to\infty$, and we deduce that case (I) holds. This completes the proof.

## III. OSCILLATION RESULTS

In this section, we establish sufficient conditions for the oscillation of all solutions of equation (1.1). We begin with the following lemma.

**Lemma 3.1.** Let $x(t)$ be a nonoscillatory solution of equation (1.1). Assume that there exists a nondecreasing function

$$\xi(t)\in C\left(\left[t_0;\infty\right)\right)$$

such that

$$\xi(t)<t \text{ and } \eta(t)=\sigma(\xi(\xi(t)))>t,\ t\geq T\geq t_0. \quad (3.1)$$

If every solution of the advanced differential equation

$$u'(t)-\left[\int_{\xi(t)}^{t}a^{-\frac{1}{\gamma}}(v)\left(\int_{\xi(v)}^{v}p(s)\left(1-c(s)\right)^{\beta}ds\right)^{\frac{1}{\gamma}}dv\right]u^{\frac{\beta}{\gamma}}\left(\eta(t)\right)=0. \quad (3.2)$$

is oscillatory, then Lemma 2.2 (I) cannot hold.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), satisfying case (I) of Lemma 2.1. We may assume that $x(t)>0$, and $x\left(\delta(t)\right)>0$, for $t\geq t_0$, (since the proof for the case $x(t)<0$ for $t\geq t_0$ is similar) then $z(t)>0$ for all $t\geq t_1\geq t_0$. from the definition of $z(t)$ and $z'(t)>0$, we have

$$x(t)\geq z(t)-c(t)z(\delta(t)) = (1-c(t))z(t). \quad (3.3)$$

It follows from equation (1.1) that

$$[a(t)[z''(t)]]'\geq p(t)(1-c(t))^{\beta}z^{\beta}(()). \quad (3.4)$$

Integrating (3.4) from $\xi(t)$ to $t$, and using the nondecreasing nature of $z(t)$ we have

$$a(t)[z''(t)]^{\gamma} - a(())[z''(())]^{\gamma}\geq z^{\beta}\left(\sigma\left(\xi(t)\right)\right)\int_{\xi(t)}^{t}(1-c(s))p(s)ds,$$

or

$$z''(t)\geq z^{\frac{\beta}{\gamma}}\left(\sigma\left(\xi(t)\right)\right)a^{-\frac{1}{\gamma}}(t)\left(\int_{\xi(t)}^{t}(1-c(s))p(s)ds,\right)^{\frac{1}{\gamma}}.$$

Integrating the last inequality again from $\xi(t)$ to $t$, we obtain

$$z'(t)\geq z^{\frac{\beta}{\gamma}}(\eta(t))\int_{\xi(t)}^{t}a^{-\frac{1}{\gamma}}(u)\left(\int_{\xi(u)}^{u}(1-c(s))p(s)ds\right)^{\frac{1}{\gamma}}du.$$

Therefore $z(t)$ is a positive solution of the inequality

$$z'(t)-\left[\int_{\xi(t)}^{t}a^{-\frac{1}{\gamma}}(u)\left(\int_{\xi(u)}^{u}(1-c(s))p(s)ds\right)^{\frac{1}{\gamma}}du\right]z^{\frac{\beta}{\gamma}}(\eta(t))\geq 0.$$

Hence, by Lemma 2.1, we conclude that the corresponding differential equation (3.2), also has a positive solution, which is a contradiction. Therefore $z(t)$ cannot satisfy case (I) of Lemma 2.2. This completes the proof.

**Lemma 3.2.** Let $x(t)$ be a nonoscillatory solution of equation (1.1). If every solution of the advanced differential equation

$$u'(t)-\left[\int_{t}^{\infty}a^{-\frac{1}{\gamma}}(v)\left(\int_{v}^{\infty}(1-c(s))p(s)ds\right)^{\frac{1}{\gamma}}dv\right]u^{\frac{\beta}{\gamma}}(\sigma(t))=0. \quad (3.5)$$

is oscillatory, then Lemma 2.2(II) cannot hold.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), satisfying case (II) of Lemma 2.2. We may assume that $x(t)>0$ and $x\left(\delta(t)\right)>0$ for all $t\geq t_0$ (since the proof for the case $x(t)<0$ for $t\geq t_0$ is similar), then $z(t)>0$ for $t\geq t_1\geq t_0$. Proceeding as in Lemma 3.1 we obtain (3.4). Integrating (3.4) from $t$ to $\infty$, and using the nondecreasing nature of $z(t)$ one gets

$$-a(t)(z''(t))^{\gamma}\geq z^{\beta}(\sigma(t))\int_{t}^{\infty}(1-c(s))^{\beta}p(s)ds.$$

Dividing the last inequality by $a(t)$ and then integrating from $t$ to $\infty$ and simplifying, we obtain

$$z'(t)\geq\left(\int_{t}^{\infty}a^{-\frac{1}{\gamma}}(u)\left(\int_{u}^{\infty}(1-c(s))^{\beta}p(s)ds\right)du\right)z^{\frac{\beta}{\gamma}}(\sigma(t)).$$

Therefore $z(t)$ is a positive solution of the differential inequality

$$z'(t)-\left(\int_t^\infty a^{-\frac{1}{\gamma}}(u)\left(\int_u^\infty (1\text{-}c(s))^\beta p(s)ds\right)du\right)z^{\frac{\beta}{\gamma}}(\sigma(t))\geq 0.$$

By Lemma 2.1, we conclude that there exists a positive solution of equation (3.5), which is a contradiction. Therefore $z(t)$ cannot satisfy case (II) of Lemma 2.2.

**Remark 3.1** It follows the proof of Lemma 3.2 that, if $x(t)$ is any nonoscillatory solution of equation (1.1) and the condition

$$\int_{t_0}^\infty p(t)dt=\infty \text{ or} \int_t^\infty a^{-\frac{1}{\gamma}}(u)\left(\int_u^\infty (1\text{-}c(s))^\beta p(s)ds\right)du=\infty$$

holds, then $z(t)$ cannot satisfy case (II) of Lemma 2.2.

**Lemma 3.3.** Let $x(t)$ be a nonoscillatory solution of equation (1.1). Suppose that there exist a function $\mu(t)\in C([t_0,\infty))$ such that

$$\tau(t)\leq\mu(t)<t. \tag{3.6}$$

If every solution of the first order delay differential equation

$$u'(t)+q(t)(1-c(t))^\alpha(\tau(t)-t_1)^\alpha\left(\int_{\tau(t)}^{\mu(t)} a^{-\frac{1}{\gamma}}(s)ds\right)^\alpha u^{\frac{\alpha}{\gamma}}(\mu(t))=0. \tag{3.7}$$

is oscillatory for all $t\geq t_1$, then Lemma 2.2 (II) cannot hold.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1) satisfying case (II) of Lemma 2.1. We may assume that $x(t)>0,\ x(\delta(t))>0$ for all $t\geq t_0$ (since the proof for the case $x(t)<0$ for $t\geq t_0$ is similar) then $z(t)>0$ for all $t\geq t_1\geq t_0$. From the definition of $z(t)$ and $z'(t)>0$, we have $x(t)\geq(1\text{-}c(t))z(t)$, and we may write the equation (1.1) as

$$(a(t)(z''(t))^\gamma)'\geq q(t)(1-c(t))^\alpha z^\alpha(\tau(t)). \tag{3.8}$$

Further

$$z(\tau(t))\geq\int_{t_1}^{\tau(t)} z'(s)ds\geq z'(\tau(t))(\tau(t)-t_1) \tag{3.9}$$

eventually. Using (3.9) in (3.8) we obtain

$$(a(t)(z''(t))^\gamma)'\geq q(t)(1-c(t))^\alpha(\tau(t)-t_1)^\alpha(z'(\tau(t)))^\alpha$$

eventually. Set $y(t)=z'(t)$, then $y(t)>0$ and

$$(a(t)(y'(t))^\gamma)'\geq q(t)(1-c(t))^\alpha(\tau(t)-t_1)^\alpha y^\alpha(\tau(t)), \tag{3.10}$$

eventually. On the other hand, by taking

$$w(t)=-a(t)(y'(t))^\gamma,$$

we see that $w(t)$ is a positive decreasing function which satisfies

$$w(\tau(t))\geq\int_{\tau(t)}^{\mu(t)}\left[-a(s)(y'(s))^\gamma\right]^{\frac{1}{\gamma}}a^{-\frac{1}{\gamma}}(s)ds$$
$$\geq w^{\frac{1}{\gamma}}(\mu(t))\int_{\tau(t)}^{\mu(t)} a^{-\frac{1}{\gamma}}(s)ds.$$

Using the last inequality in (3.10), we see that $w(t)$ is a positive solution of the differential inequality

$$w'(t)+q(t)(1-c(t))^\alpha(\tau(t)-t_1)^\alpha\left(\int_{\tau(t)}^{\mu(t)} a^{-\frac{1}{\gamma}}(s)ds\right)^\alpha w^{\frac{\alpha}{\gamma}}(\mu(t))\leq 0.$$

By Theorem 1 in [9], we see that the corresponding differential equation (3.7) also has a positive solution, a contradiction. Therefore $z(t)$ cannot satisfy case (II) of Lemma 2.2.

**Lemma 3.4.** Let $x(t)$ be a nonoscillatory solution of equation (1.1). Assume that

$$\mathrm{a(t)}\in \mathrm{C}^1([\mathrm{t}_0,\infty)),\ \mathrm{a}'(\mathrm{t})\geq 0, \text{ and } \gamma\neq 1. \tag{3.11}$$

If every solution of the first order delay differential equation

$$u'(t)+q(t)(\tau(t)\text{-}t_1)(1\text{-}c(t))^\alpha(t\text{-}\tau(t))a^{-\frac{\alpha}{\gamma}}(t)u^{\frac{\alpha}{\gamma}}(\tau(t)) = 0 \tag{3.12}$$

is oscillatory for $t\geq t_1$, then Lemma 2.2 (II) cannot hold.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1) satisfying case (II) of Lemma 2.2. We may assume that $x(t)>0,\ x(\delta(t))>0$ for all $t\geq t_0$ (since the proof for the case $x(t)<0$ for $t\geq t_0$ is similar). Then $y(t)=z'(t)$ satisfies (3.10). From (3.11) and

$$0<(a(t)(y'(t))^\gamma)'=a'(t)(y'(t))^\gamma+a(t)\gamma[y'(t)]^{\gamma-1}y''(t)$$

we get $y''(t)<0.$ Hence

$$\mathrm{y}(\tau(\mathrm{t}))\geq\int_{\tau(t)}^t -y'(s)ds\geq -y'(\tau(t))(t-\tau(t)). \tag{3.13}$$

Using (3.13) in (3.10), we see that $w(t)=-a(t)(y'(t))^\gamma$ is a positive solution of the differential inequality

$$w'(t)+q(t)(\tau(t)\text{-}t_1)(1\text{-}c(t))^\alpha(t\text{-}\tau(t))a^{-\frac{\alpha}{\gamma}}(t)w^{\frac{\alpha}{\gamma}}(\tau(t))\leq 0.$$

It follows from Theorem 1 in [9], we see that the differential equation (3.12) also has a positive solution, a contradiction. Hence $z(t)$ cannot satisfy case (II) of Lemma 2.2.

Next we establish some new oscillation criteria for equation (1.1).

**Theorem 3.5.** Assume that one of the following conditions is satisfied:

($H_1$) Every solution of the equation (3.5) is oscillatory.

($H_2$) Condition (3.6) holds and every solution of the equation is (3.7) oscillatory for $t \geq t_1$ large enough.

($H_3$) Condition (3.11) holds and every solution of the equation (3.12) oscillatory for $t \geq t_1$ large enough.

Then every solution of equation (1.1) is either oscillatory or

$\lim_{t\to\infty}|x(t)| = \infty$, and further there exists a constant $d > 0$

$$|x(t)| \geq d\int_{t_1}^{t}\frac{(t-s)}{a^{\frac{1}{\gamma}}(s)}ds. \tag{3.14}$$

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $x(t) > 0$ and $x(\delta(t)) > 0$ for all $t \geq t_1 \geq t_0$ (since the proof for the case $x(t) < 0$ is similar). Then $z(t)$ satisfies either case (I) or (II) of Lemma 2.2. But in view of Lemmas 3.2, 3.3 and 3.4 and by the conditions ($H_1$), ($H_2$) and ($H_3$) respectively, we see that the case (II) cannot hold. Therefore $z(t)$ satisfies case (I) of Lemma 2.2, which implies that there is a constant $M > 0$ such that $a(t)(z''(t))^{\gamma} \geq M$. Integrating twice from $t_1$ to $t$ we have

$$z(t) \geq M^{\gamma}\int_{t_1}^{t}\frac{(t-s)}{a^{\frac{1}{\gamma}}(s)}ds. \tag{3.15}$$

From the definition of $z(t)$, we have

$$x(t) \geq (1-c(t))z(t). \tag{3.16}$$

Combining (3.15) and (3.16) we obtain (3.14). This completes the proof.

**Remark 3.2.** Theorem 3.5 provides three criteria for the oscillation of all solutions of equation (1.1) and these criteria involve both delayed and advanced argument. Thus, Theorem 3.5 extends and complements other known criteria.

In view of Lemma 3.1, one can establish new criteria for equation (1.1)

**Theorem 3.6.** Assume that one of the conditions ($H_1$)-($H_3$) is satisfied. Further assume that ($H_4$) Conditions (3.1) holds and every solution of equation (3.2) is oscillatory. Then every solution of equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Then $z(t)$ is also nonoscillatory and it satisfies case (I) or case (II) of Lemma 2.2. But case (II) is eliminated due to Lemmas 3.2 to 3.4 and case (I) is excluded by Lemma 3.1. This completes the proof.

By using criteria for the oscillation of equations (3.2) and (3.5), we immediately obtain sufficient conditions for the oscillation of all solutions of equation (1.1).

**Theorem 3.7.** Let $\beta = \gamma$ holds. Assume that

($A_1$) conditions (3.1) and

$$\liminf_{t\to\infty}\int_t^{\eta(t)}\left[\int_{\xi(v)}^{v} a^{-\frac{1}{\gamma}}(s)\left(\int_{\xi(s)}^{s} p(u)du\right)^{\frac{1}{\gamma}} ds\right]dv > \frac{1}{e(1-c)},$$

($A_2$) $$\liminf_{t\to\infty}\int_t^{\sigma(t)}\left[\int_v^{\infty} a^{-\frac{1}{\gamma}}(s)\left(\int_s^{s} p(u)du\right)^{\frac{1}{\gamma}} ds\right]dv > \frac{1}{e(1-c)},$$

are satisfied. Then every solution of equation (1.1) is oscillatory.

Proof. It follows from Theorem 2.3 in [6] that conditions ($A_1$) and ($A_2$) imply the oscillation of all solutions of equations (3.2) and (3.5) respectively. Then the conclusion follows from Theorem 3.6.

If we omit condition ($A_1$) in Theorem 3.7, then we obtain the following result.

**Theorem 3.8.** Assume that condition (A2) holds. Then every solution of equation (1.1) is either oscillatory or $\lim_{t\to\infty}|x(t)| = \infty.$

## IV. EXAMPLES

In this section we present some examples to illustrate the main results.

**Example 4.1.** Consider the third order neutral differential equation with mixed arguments

$$\left(t^3\left[\left(x(t)+\frac{1}{2}x(t-1)\right)''\right]^3\right)' = \frac{2}{t^{\frac{1}{2}}}x^3(t/2)+\frac{3}{t^{\frac{1}{3}}}x^3(4t),\ t\geq 1. \tag{4.1}$$

We let $\xi(t) = 3t/4$ then it is easy to see that all conditions of Theorem 3.7 are satisfied, and hence all solutions of equation (4.1) are oscillatory.

**Examples 4.2**. Consider the neutral differential equation

$$\left(t^3\left[\left(x(t)+\frac{1}{3}x(t-2)\right)''\right]^3\right)' = \frac{2}{t^{\frac{1}{2}}}x^3(t/2)+\frac{3}{t^{\frac{1}{3}}}x^5(2t),\ t\geq 1. \tag{4.2}$$

By Theorem 3.8, every solution of equation (4.2) is either oscillatory or $|x(t)| \to \infty$ as $t \to \infty$.

## V. CONCLUSION

In this paper, we have established some oscillation theorems for equation (1.1) by comparing the oscillation of a set of suitable first delay / advanced type differential

equations. So, investigation of oscillatory behavior of third order neutral differential equations is reduced to that of first order differential equations. It would be interesting to investigate the oscillatory behavior of equation (1.1) when $\int_{t_0}^{\infty} a^{-\frac{1}{\gamma}}(t)dt < \infty,$ and the details are left to the reader.

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